Limits of functions

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Plan

- Continue on limits of functions
- 10 min break
- Continuous functions

Excises

Find the following limits:

$$\lim_{x\to 1} 5x^3 + x^2 + 1$$

$$\lim_{x\to 3} \frac{x+2}{x^2+1}$$

$$\lim_{x\to 0} (\frac{1}{x} - \frac{1}{x+1})$$

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$$

Sandwich Theorem for Functions

Theorem

If $f(x) \leqslant g(x) \leqslant h(x)$ *for all* $x \in \mathbb{R} \backslash \{x_0\}$ *and*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L,$$

then $\lim_{x \to x_0} g(x) = L$.

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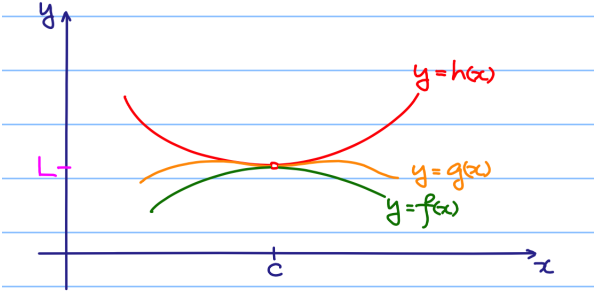

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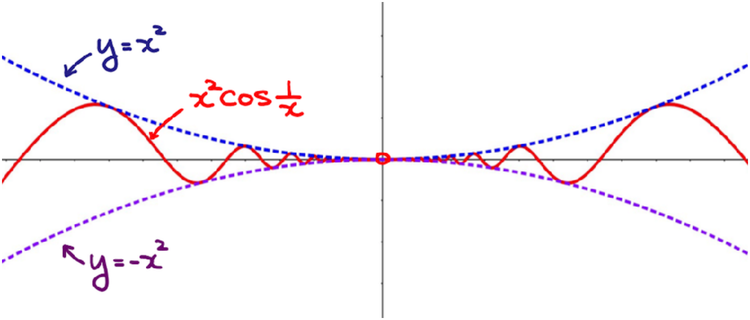

Limits at infinity of functions

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Definition

Let $f:\mathbb{R}\to\mathbb{R}$ be a function. If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger, then L is called the limit of $f(x)$ at positive $+\infty$. We write

$$\lim_{x\to+\infty} f(x) = L.$$

$$x\to+\infty \quad \Longrightarrow \quad f(x)\to L.$$

Limits at infinity

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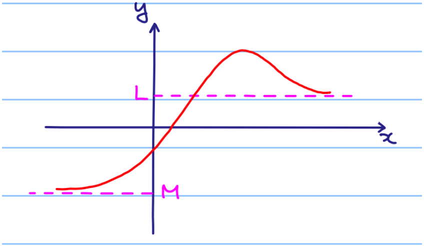

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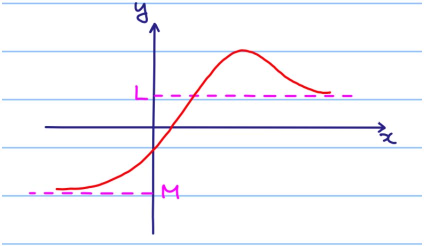

From this picture we could write

$$\lim_{x \to +\infty} f(x) = L \quad \text{and} \quad \lim_{x \to -\infty} f(x) = M.$$

Example

Find $\lim_{x\to+\infty}\frac{1}{x}$, $\lim_{x\to-\infty}\frac{1}{x}$, $\lim_{x\to 1}\frac{1}{x}$ and $\lim_{x\to 0}\frac{1}{x}$.

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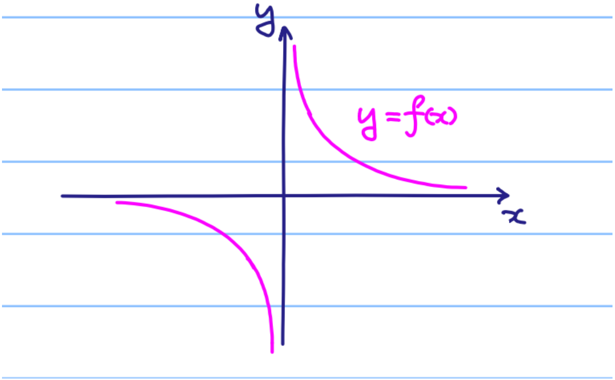

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In fact we also have

$$\lim_{x\to+\infty}\left(\frac{1}{2}\right)^x = 0 \quad \lim_{x\to-\infty} 2^x = 0.$$

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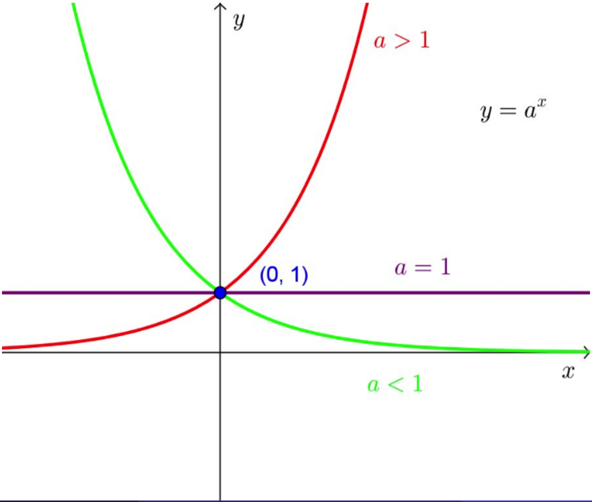

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$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}, \quad \text{if} \quad \lim_{x\to+\infty} g(x) \neq 0.$$

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$$= \frac{3}{1+0+0} = 3.$$

Rational polynomials

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Let $p(x)$ and $q(x)$ are polynomials that
$p(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$ with $a_m \neq 0$ and
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$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} = \begin{cases} \frac{a_m}{b_n} & \text{if } m = n \\ 0 & \text{if } m < n \\ \pm\infty & \text{if } m > n \end{cases}.$$

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(1) Let $p(x)$ be a polynomial then for any $a > 1$ we have

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Ok, let's have 10 minutes break!